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Covering with latin transversals

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Abstract

Given an $n \times n$ matrix $A = [a_{ij}]$, a *transversal* of A is a set of elements, one from each row and one from each column. A transversal is a *latin transversal* if no two elements are the same. Erdös and Spencer showed that there always exists a latin transversal in any $n \times n$ matrix in which no element appears more than s times, for $s \leqslant (n-1)/16$. Here we show that, in fact, the elements of the matrix can be partitioned into n disjoint latin transversals, provided n is a power of 2 and no element appears more than εn times for some fixed $\varepsilon > 0$. The assumption that n is a power of 2 can be weakened, but at the moment we are unable to prove the theorem for all values of n.

1. Introduction

Given an $n \times n$ matrix $= [a_{ij}]$, a *transversal* of A is a set of elements, one from each row and one from each column. A transversal is a *latin transversal* if no two elements are the same. There have been more conjectures than theorems on latin transversals in the literature. Erdös et al. [5] provide a good overview of the work done on transversals. Previous work can be found also in [10, 4].

Recently, Erdös and Spencer [7] showed that there always exists a latin transversal in any $n \times n$ matrix in which no element appears more than s times, for $s \leqslant (n-1)/16$. Here we show that, in fact, all the elements of the matrix can be partitioned into latin transversals, provided n is a power of 2 and no element appears more than εn times for some fixed $\varepsilon > 0$.

Theorem 1. *Let n be 2^m. Any $n \times n$ matrix in which no element appears more than s times contains n disjoint latin transversals provided $s \leqslant \varepsilon n$ (for ε, an absolute constant $\ll 1$).*

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The assumption that n is a power of 2 can be weakened, but at the moment we are unable to prove the theorem for all values of n. On the other hand, our proof can be easily modified to prove the existence of many pairwise disjoint transversals in any n by n matrix in which no entry appears more than εn times, without any restriction on n. Therefore, our method implies a strengthening of the result of [7] for any n (apart from the actual value of the constant ε). We prove the above theorem in a more general framework, stated in terms of graphs.

Let n and M denote positive integers. Let $G = (V, E)$ denote a graph where $V = \{(i, j, k)\colon 1 \leqslant i, j \leqslant n,\ 1 \leqslant k \leqslant M\}$. It is convenient to imagine the vertices partitioned into M *blocks*, where each block has n^2 vertices, arranged in an $n \times n$ array. Thus, the index k denotes which block a vertex (i, j, k) belongs to, and the indices i, j identify the vertex within a block.

Definition. *$U \subseteq V$ is a generalized diagonal* if (i) $\forall i, k$ there exists a unique j such that $(i, j, k) \in U$ and (ii) $\forall j, k$ there exists a unique i such that $(i, j, k) \in U$.

Theorem 2. *Let $n = 2^m$ and let $G = (V, E)$ be as defined above, and suppose the maximum degree of G is at most εn, where ε is a small absolute constant (any $\varepsilon \leqslant 10^{-10^{10}}$ will do). Then there exists a proper coloring $f\colon V \to \{1, \ldots, n\}$ such that $\{(i, j, k)\colon f(i, j, k) = \alpha\}$ is a generalized diagonal, for all $1 \leqslant \alpha \leqslant n$.*

The proof of the above theorem is probabilistic, the main tool being the Lovász Local Lemma which can be stated as follows.

Lemma 1 (The local lemma (Erdös and Lovász [6]). *Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose each A_i is mutually independent of all but at most b other events A_j and suppose the probability of each A_i is at most p. If $ep(b + 1) < 1$ then with positive probability none of the events A_i holds.*

The proof can also be found in e.g. [9, 2]. Also crucial to our proof is the following result of the first author.

Theorem 3 (Alon [1]). *There exists an absolute positive constant c such that for any two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices, where G_1 has maximum degree d, and G_2 is a vertex disjoint union of cliques of size cd each, the chromatic number of the graph $G = (V, E_1 \cup E_2)$ is precisely cd.*

Proof of Theorem 1 (*from Theorem* 2). Theorem 1 follows easily from Theorem 2. Given an $n \times n$ matrix, we can associate the following graph with it. The vertices correspond to the elements of the matrix, and there is an edge between two vertices whenever the elements have the same value. Observe that a generalized diagonal of this graph corresponds to a transversal in the matrix. Thus, a proper coloring as in Theorem 2 supplies the existence of the desired n disjoint latin transversals. □

The proof of Theorem 2 is rather complicated and is described in the next two sections.

2. An outlined proof of Theorem 2

We divide the proof into four steps. Let us denote the claim in the theorem by the problem instance $(G, n, M, \varepsilon n)$. The following is an outline of the four steps of the proof. The details are provided in the next section.

Step 1: The aim of this step is to divide the problem into subproblems so that the degree of each vertex in each subproblem is much smaller. Randomly split each $n \times n$ block into four blocks of size $n/2 \times n/2$, say, by randomly splitting the rows into two parts, and the columns into two parts. The "diagonal" two blocks form one graph and the other two blocks of another graph. (Thus, with each such split both the number of graphs and the number of blocks in each graph is doubled.) Repeat this random splitting until n becomes $X = (1/\varepsilon)^{1+\delta}$ (where δ is some small fixed positive constant), and M becomes $M \cdot n/X$, so that the maximum degree of each vertex in the induced subgraph of G on the set of vertices in its part becomes at most $2\varepsilon X \leqslant 2(1/\varepsilon)^{\delta}$. Thus, at the end of this step we are left with n/X subproblems of type $(G', X, Mn/X, 2\varepsilon X)$.

Step 2: Randomly split each block (of size $X \times X$) further into subblocks of size $Y \times Y = X^{0.01}$ so that the total number of edges in each induced subgraph G', which are not vertical or horizontal, and are incident with any given subblock is at most $Y^{0.01}$. Thus, in this step we restrict the total degree of each subblock of vertices.

Step 3 (Final partition): Randomly split each subblock into *small blocks* of size $Z \times Z = Y^{0.1}$, so that each small block is an independent set (besides, possibly the vertical or the horizontal edges, which we once again ignore).

Step 4: Partition each small block of size $Z \times Z$ into Z transversals, arbitrarily. Define two graphs on the same vertex set, the vertices representing all these transversals, and the edges defined as follows. In the first graph there is an edge between two vertices if and only if the corresponding transversals belong to the same small block; in the second graph, an edge denotes that there is an edge of G between a member of the first transversal and a member of the second. The proof is completed by invoking Theorem 3.

3. The details

In order to make the presentation more coherent we do not use the integer signs $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ in this section and assume that all quantities appearing here are integers. Since we may assume (by choosing a sufficiently small $\varepsilon > 0$) that the numbers we deal with are sufficiently large this assumption is justified. We also assume, whenever it is needed, that the number of vertices in our graph and the bound we have for its maximum degree are sufficiently large.

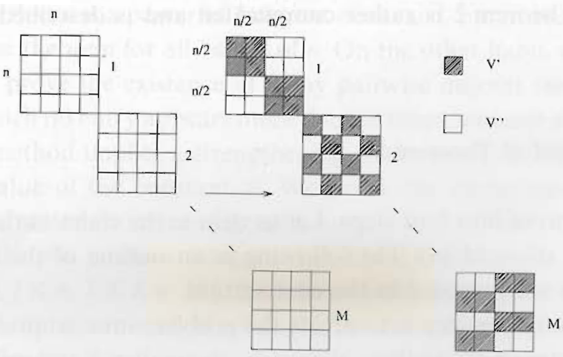

Fig. 1.

Step 1: Given a graph $G = (V, E)$ whose set of vertices V is the one described above, let us identify the blocks of vertices as follows. Put $V = V_1 \oplus V_2 \oplus \cdots \oplus V_M$, where $V_p = \{(i, j, p): i, j \in N = \{1, \ldots, n\}\}$. (Here and in what follows we use the symbol $\oplus$ to denote disjoint union.) Suppose the maximum degree of G is at most d. The following lemma is the basic tool in the repeated partitioning that we perform in this step. (Recall that we assume that the size n of each block is a power of 2 and is thus even.)

Lemma 2. *For every positive (small) constant $\gamma > 0$ there is a (large) $C_1 = C_1(\gamma)$ such that if $n > C_1$, $d \geqslant n^\gamma$ and G is as above, then there exists a partition $V = V' \oplus V''$ with the following properties.*

(a) $\forall p, 1 \leqslant p \leqslant M, \ \exists U_p, W_p \subseteq N, \ |U_p| = |W_p| = n/2, \ V' \cap V_p = \{(i, j, p): (i \in U_p$ *and* $j \in W_p) \cup (i \in N \setminus U_p$ *and* $j \in N \setminus W_p)\}$ *and*

(b) *if $G' = G[V']$, $G'' = G[V'']$ are the induced subgraphs of G on V' and on V'', respectively, then the maximum degree of G' as well as that of G'' are both at most $(d/2) + d^{4/5}$.*

Proof. The method resembles the one used in [1] but some new ideas have to be incorporated. For each p, let U_p and W_p be random subsets of N of cardinality $n/2$ each, chosen uniformly and independently, and let $V = V' \oplus V''$ be the corresponding partition of V (see Fig. 1). For each vertex $v = (i, j, k)$, define A_v to be the event that the degree of v in its graph (G' if $v \in V'$ or G'' if $v \in V''$) is bigger than $(d/2) + d^{4/5}$. For each p, $1 \leqslant p \leqslant M$, let B_p be the (bad) event that an event A_v occurred for some vertex v in block number p. Note that each event B_p is independent of all other events but those that correspond to blocks whose *distance* from block number p is at most 2. Here we define the *distance* between blocks as the distance between them in the graph whose vertices are the blocks in which two are adjacent iff there is an edge of G connecting two members of these two blocks. It follows that each event B_p is independent of all but at most $b = n^2 d + (n^2 d)^2 < n^7$ other events.

Claim 1. $\Pr[A_v] < e^{-cd^{0.1}}$, *where c is some absolute constant.*

Proof. We can clearly assume that v has exactly d neighbors. Partition the set of d neighbors of v into at most $3\sqrt{d}$ sets of size at most $\sqrt{d}$ each, such that for each set either (a) all are in a row, or (b) all are in a column, or (c) no two are in a row or in a column. To show that this is possible we argue as follows. First we show that we can partition any set of d entries in a matrix into at most $g(d) \leqslant 2\sqrt{d}$ subsets (of any size), each satisfying either (a) or (b) or (c). This is certainly true for $d = 1$. Assuming it holds for all $d' < d$ we prove it for d, $d > 1$. By the Hall–König theorem [8] either there are at least $2\sqrt{d} - 1$ entries on a diagonal (i.e. no two in a row or in a column), or at most $2\sqrt{d}$ lines suffice to cover all these entries. In the second case we are done, and in the first case we conclude, by the induction hypothesis, that

$$g(d) \leqslant 1 + g(d - 2\sqrt{d} + 1) \leqslant 1 + 2\sqrt{d - 2\sqrt{d} + 1} \leqslant 2\sqrt{d}.$$

Thus, we have at most $2\sqrt{d}$ sets, each of type (a) or (b) or (c). Now break each set with more than $\sqrt{d}$ elements into sets of size $\sqrt{d}$ each and, possibly, one smaller set of the remaining elements. At the end of this process we obtain at most $2\sqrt{d}$ of these smaller sets, since each of our original sets can contribute at most one such set. The total number of sets of size $\sqrt{d}$ we can have is clearly at most $\sqrt{d}$, and thus the desired partition exists (and in fact the constants can be slightly improved). Without loss of generality, suppose that $v \in V'$. Consider, now, a specific subset A among the ones in the partition of the set of all neighbors of v described above, and let a denote the number of its elements, $a \leqslant \sqrt{d}$. Since a is much smaller than n, and since A satisfies (a), (b) or (c), it is not too difficult to check that the random variable that counts the number of members of A that lie in V' is well approximated by a Binomial random variable with parameters a and $\frac{1}{2}$.

It thus follows, from the standard estimates for Binomial distribution (see, e.g. [2]) that the probability that this random variable exceeds $a/2 + d^{4/5}/3\sqrt{d}$ is at most

$$e^{-c'(d^{0.3})^2/a}$$

for some absolute constant $c' > 0$. Since $a \leqslant \sqrt{d}$ the last probability is bounded by $e^{-c'd^{0.1}}$. Observe, now, that if A_v occurred then there must be a set A in the partition with more than $|A|/2 + d^{4/5}/3\sqrt{d}$ of its members chosen to V'. Therefore, the probability of the event A_v is at most $3\sqrt{d}e^{-c'd^{0.1}} \leqslant e^{-cd^{0.1}}$ for some $c > 0$. This completes the proof of the claim. □

Proof of Lemma 2 (*conclusion*). Observe that by Claim 1 the probability of each event B_p is at most $n^2 e^{-cd^{0.1}}$. By our assumption $d \geqslant n^\gamma$ and $n > C_1(\gamma)$. Hence, the probability of each B_p is much smaller than, say, $1/n^8$. As observed above, each B_p is mutually independent of all but less than n^7 other events $B_{p'}$, and thus, by the local lemma (Lemma 1), the assertion of Lemma 2 follows. □

Repeated Partitioning. To complete Step 1, apply Lemma 2 repeatedly until X becomes $(1/\varepsilon)^{1+\delta}$, where $\delta > 0$. Note that we get independent problems of type $(G^{(l)}, X, M \cdot n/X, d(X))$, for $l = 1, \dots, n/X$, where $d(X)$ is an upper bound on the maximum degree of all the induced graphs $G^{(l)}$. We next bound this maximum degree $d(X)$ after the repeated partitioning.

Claim 2. *$d(X) \leqslant 2\varepsilon X$ as long as $X \geqslant (1/\varepsilon)^{1+\delta}$ and $\varepsilon \leqslant \varepsilon_0(\delta)$.*

Proof. We start with blocks of size $n_0 = n$, and maximum degree $d_0 = \varepsilon n$. We then apply Lemma 2 repeatedly, noting that its assertion holds as long as the bound for our degree d_i is at least n_i^γ and $n_i > C_1(\gamma)$. In each application of Lemma 2 we achieve

$$n_{i+1} = \frac{n_i}{2}, \qquad d_{i+1} \leqslant \frac{d_i}{2} + d_i^{4/5} \quad \Rightarrow \quad n_i = \frac{n}{2^i};$$

define $z_i = d_i^{1/5}$ so the recurrence becomes

$$z_{i+1}^5 \leqslant \frac{z_i^5 + 2z_i^4}{2} \leqslant \frac{(z_i+1)^5}{2} \quad \Rightarrow \quad z_{i+1} \leqslant \frac{z_i + 1}{2^{1/5}}.$$

Define $t_i = z_i + c$ so that c satisfies $c = (c-1)/2^{1/5}$. Then $t_{i+1} \leqslant t_i/2^{1/5}$ and hence

$$t_i \leqslant \frac{t_0}{2^{i/5}} \quad \Rightarrow \quad z_i \leqslant \frac{z_0 + c}{2^{i/5}} - c.$$

$$d_i \leqslant \left(\frac{d_0^{1/5} + c}{2^{i/5}} - c\right)^5 \leqslant \frac{(d_0^{1/5} + 10)^5}{2^i}.$$

Therefore,

$$d_i \leqslant \frac{d_0}{2^i} + \mathrm{O}\left(\frac{d_0^{4/5}}{2^i}\right) \leqslant \frac{2d_0}{2^i},$$

where the last inequality holds since in our range $d_0/2^i$ is sufficiently large. If we iterate till n_i becomes $X = (1/\varepsilon)^{1+\delta}$, then $d_i \leqslant 2\varepsilon X$ and hence the claim. □

Step 2: At this stage we have a graph G' consisting of blocks of vertices of size $X \times X$ each, and the maximum degree is at most $2\varepsilon X \leqslant X^\gamma$. Note that by the analysis in Step 1, γ can be chosen to be an arbitrarily small positive constant provided ε is sufficiently small.

For each block independently and randomly, partition the rows into $R_1 \oplus \cdots \oplus R_r$, and the columns into $C_1 \oplus \cdots \oplus C_r$, where $r = X^{0.99}$. Thus, we get a chain of *subblocks* (of vertices), each of size $Y \times Y$ where $Y = X^{0.01}$. More precisely, the vertex set of this (say, the ith) chain is $R_j \times C_l$, such that $j + l \equiv i \pmod r$.

We want to prove, using the local lemma, that with positive probability the number of edges incident with each subblock in the subgraph induced on its chain is small ($< X^{0.0001}$). For each subblock q, define A_q to be the event that the subblock has *too*

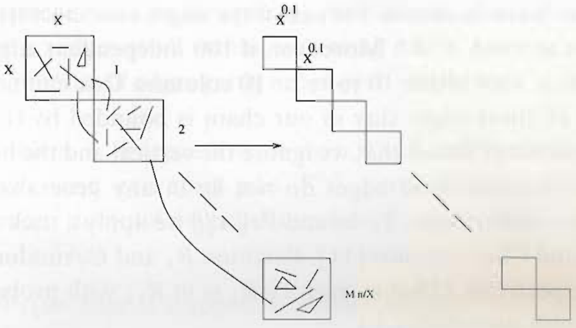

Fig. 2.

many ($> X^{0.0001}$) edges. Clearly, each event A_q is independent of all other events but those corresponding to subblocks from blocks whose *distance* from the block of q is at most 2. Here we define, as before, the *distance* between blocks as the distance between them in the graph whose vertices are the blocks and two are adjacent if there is an edge of G' joining a member of the first block with one of the second. Thus, each event is independent of all others but at most b, where

$$
\begin{aligned}
b &\leqslant \text{(number of blocks of distance 2)} \cdot \text{(number of subblocks in each block)} \\
&\leqslant [1 + X^2 \cdot X^{\gamma} + (X^2 \cdot X^{\gamma})^2] \cdot (X^{0.99})^2 \\
&\leqslant X^6.
\end{aligned}
$$

We now want to bound $\Pr[A_q]$. Let A'_q be the event that the number of edges from q to vertices outside q is $\geqslant X^{0.0001}/2$, and let A''_q be the event that the number of edges within q is $\geqslant X^{0.0001}/2$. Clearly,

$$
\Pr[A_q] \leqslant \Pr[A'_q] + \Pr[A''_q].
$$

We bound the left-hand side by bounding, separately, the two probabilities on the right-hand side. Assume the subblock we are dealing with is $R_1 \times C_1$, without loss of generality.

Part A (edges between q and vertices outside it).

$$
\begin{aligned}
\Pr[A'_q] &\leqslant \Pr[\text{there exists at least } 200X^{\gamma} \text{ edges going out of } q] \\
&\leqslant \Pr[\text{there exists 100 independent edges going out of } q] \\
&\leqslant \binom{Y^2 X^{\gamma}}{100} \cdot \left(\frac{1}{X^{0.99}} \right)^{10} \\
&\leqslant \frac{X^{2+100\gamma}}{X^{9.9}} < \frac{1}{X^7}.
\end{aligned}
$$

This is because we have to choose 100 out of the edges incident with the vertices of q, whose number is at most Y^2X^γ. Moreover, if 100 independent edges leave out of q, they must end on at least either 10 rows or 10 columns. One can now check that the probability that all these edges stay in our chain is bounded by $(1/X^{0.99})^{10}$.

Part B (*edges within q*). Recall that we ignore the vertical and the horizontal edges in this step. This is because these edges do not lie in any generalized diagonal, and cannot cause any problem later. To bound $\Pr[A_q'']$ we apply a trick similar to the one used by Vapnik and Chervonenkis [11]. Partition R_1 and C_1 randomly into R_{11}, R_{12} and C_{11}, C_{12}, respectively. (That is, each $r \in R_1$ is in R_{11} with probability $\frac{1}{2}$ and each $c \in C_1$ is in C_{11} with probability $\frac{1}{2}$.)

Define A_q''' to be the event that A_q'' occurred AND that at least a tenth of the edges within the same subblock q "cross", i.e. go either from $R_{11} \times C_{11}$ to $R_{12} \times C_{12}$ or from $R_{12} \times C_{11}$ to $R_{11} \times C_{12}$.

Claim 3. $\Pr[A_q'''] \geqslant \Pr[A_q''] \cdot \frac{1}{6}$.

Proof. Note that

$$\Pr[A_q'''] = \Pr[A_q''] \cdot \Pr[\geqslant \tfrac{1}{10} \text{ edges "cross"} \mid A_q''].$$

The expected number N_σ of edges that cross is $\frac{1}{4}$ times the total number of edges within the subblock.

If the probability that at least a tenth cross is p, then

$$p \cdot 1 + (1-p) \cdot \tfrac{1}{10} \geqslant \tfrac{1}{4} \Rightarrow p \geqslant \tfrac{1}{6}. \qquad \square$$

On the other hand,

$$\Pr[A_q'''] \leqslant \Pr[\geqslant \tfrac{1}{20} X^{0.0001} \text{ edges cross}].$$

Choose first R_{11}, C_{11} and then R_{12}, C_{12} from the total $X \times X$ block. Then the same computation as before (Part A) shows that the probability to get 100 independent edges is $< 2/X^7$. Hence, by Claim 3, we have, $\Pr[A_q''] \leqslant 6 \Pr[A_q'''] \leqslant 12/X^7$.

This completes the proof of Part B. From Parts A and B we conclude that

$$\Pr[A_q] \leqslant 13/X^7.$$

Since we saw that each event is independent of all but at most X^6 others, we can conclude using the local lemma, that there exists a way of partitioning the $X \times X$ blocks into subblocks of size $X^{0.01} \times X^{0.01}$ so that each subblock has at most $X^{0.0001}$ edges incident with it. This completes the proof of Step 2.

Step 3: We now have subblocks of size $Y \times Y$ such that the number of edges incident with vertices in each subblock is $\leqslant Y^{0.01}$. (Note that these edges may be within the same subblock or between the subblock in consideration and another subblock.)

We want to achieve, via a final partition, *small blocks* of size $Z \times Z$ where $Z = Y^{0.1}$ so that there are no edges at all within each small block. Although we do not care (in this step) for the number of edges going out of each small block, we know it is, trivially, $\leqslant Y^{0.01} \leqslant Z^{0.1}$.

Thus, the problem is independent in each $Y \times Y$ subblock. Consider a random partition of rows (and columns) into $Y^{0.9}$ parts, yielding small blocks of size $Y^{0.1} \times Y^{0.1}$. Denote the total expected number of edges within all the $(Y^{0.9})^2$ small blocks by $E(Y, Z)$. Then

$$E(Y,Z) \leqslant \Pr[\text{an edge is trapped in a small block}]$$
$$\times \text{number of edges incident with a subblock}$$
$$\leqslant \frac{1}{(Y^{0.9})^2} \times Y^{0.01} \ll 1.$$

(Note that we used the fact that there are no vertical or horizontal edges, but in fact we do not have to assume it here because $1/(Y^{0.9}) \times Y^{0.01}$ is still $\ll 1$.)

Thus, there exists a way of partitioning so that each small block of size $Z \times Z$ has no edges inside it.

Step 4: At this state, we have n/Z chains of $Mn/Z = M'$ (say) small blocks of size $Z \times Z$ each, and each small block is an independent set. Furthermore, the number of edges out of each small block to its chain is at most $Z^{0.1}$. The idea now is to find *subdiagonals* in each small block, and claim that we can put the subdiagonals together to form Z disjoint generalized diagonals for the original problem. (All these diagonals arising from all n/Z chains supply the n generalized diagonals in the assertion of Theorem 2.) We achieve this using Theorem 3 as follows.

We partition each $Z \times Z$ block into Z diagonals arbitrarily! Define two graphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ on these diagonals as follows. Let $V = \{D_i^j, 1 \leqslant j \leqslant M', 1 \leqslant i \leqslant Z\}$, be the set of diagonals from all the small blocks. In H_1, there is an edge between vertices representing diagonals from the same $Z \times Z$ block. Thus, E_1 is a disjoint union of M' cliques, each of size Z. In H_2, there is an edge between D_i^j and $D_{i'}^{j'}$ if and only if there is an edge of G between a member of the first diagonal and a member of the second.

Note that the maximum degree in H_2 is $\leqslant Z^{0.1}$, since that is the maximum number of edges out of any small block. By Theorem 3, we know we can properly color the vertices of $H = (V, E_1 \cup E_2)$ using Z colors. But each color class corresponds to a set of *subdiagonals* that is independent in G. This completes the proof.

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